A Note on the Mixing Length Theory of Turbulent Flow

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In 1925 Prandtl proposed the mixing length theory of turbulent flow by analogy with the kinetic theory of gases so that the Reynold's shear stress term $-\rho\,\overline{u'v'}$ can be expressed as (details can be found in references *1* to *7*)

$$-\overline{u'v'} = l_p^2 \left(\frac{d\bar{u}}{dy}\right)^2 \tag{1}$$

Although it is now accepted that the concept of a mixing length does not adequately represent the correct physical picture of the structure of turbulence in detail, mixing length theories are nevertheless most useful to engineers as a means of correlating and extrapolating experimental data. The reason for this, as discussed by Hinze (*3*), is simply that more correct theories that can be used successfully from a practical engineering point of view are not available.

The purpose of this note is to show that the mixing length theory can be made more useful if it is reformulated and some elementary mathematical properties of the result are recognized.

Let us first briefly discuss some of the limitations of Prandtl's mixing length theory. First, it assumes that a lump of fluid retains its identity over a certain distance after which it loses its momentum to the surroundings and adopts the properties of its environment. This oversimplified picture fails to give physical insight into the structure of turbulent flow because no attempt is made to explain how and why a fluid lump will retain its identity and the mechanism by which it will adopt the properties of the surrounding. It assumes that the mixing length and eddy diffusion depend only on local conditions in the flow. However, it can be seen from the experimental investigations of Clauser (*8*) and Corino and Brodkey (*9*), among others, that eddies originate in a region near the wall ($y^+ < 70$). These eddies then move towards the turbulent core region and are at the same time convected downstream. Consequently, the turbulent core region ($y^+ > 70$) contains eddies which originated within the generation region ($y^+ < 70$) at various upstream positions. But this assumption will not be a serious limitation if the flow is axially homogeneous, for example, as in the case of fully developed turbulent flow in a pipe. Second, it assumes that the diffusion and convection of turbulent energy are negligibly small, so that the turbulent energy generated locally is equal to the dissipation. However, it has been found (*3*) that the diffusion and convection terms in the energy equation are, in general not negligible. Third, it assumes that the rate of transport is proportional to the gradient of the mean velocity. Therefore, a coefficient of eddy diffusion ϵ can be introduced as

$$-\overline{u'v'} = \epsilon \frac{d\bar{u}}{dy} \tag{2}$$

and, from Equation (1), we have

$$\epsilon = l_p^2 \left(\frac{d\bar{u}}{dy}\right) \tag{3}$$

The use of Equation (3) implies that momentum is transferred mainly by small eddies, and it will be shown later that this assumes

$$l_p << 2 \left| \frac{d\bar{u}/dy}{d^2\bar{u}/dy^2} \right| \tag{4}$$

Equation (3) predicts that ϵ goes to zero whenever the velocity gradient vanishes. This limits the practical application of the mixing length theory and the eddy diffusion concept, but this difficulty will be overcome in the present study. As a matter of fact, Prandtl (*1*) realized this limitation, and to eliminate it he proposed the following modification:

$$-\overline{u'v'} = l_p^2 \left(\frac{d\bar{u}}{dy}\right) \sqrt{\left(\frac{d\bar{u}}{dy}\right)^2 + l_{p1}^2 \left(\frac{d^2\bar{u}}{dy^2}\right)^2}$$

where l_{p1} is an additional unknown which, again, must be obtained from experimental measurements. The expression for eddy diffusivity ϵ, Equation (2), according to this more general assumption, becomes

$$\epsilon = l_p^2 \sqrt{\left(\frac{d\bar{u}}{dy}\right)^2 + l_{p1}^2 \left(\frac{d^2\bar{u}}{dy^2}\right)^2}$$

This equation complicates the computations considerably and consequently has been used only occasionally.

Our purpose here is to reformulate the mixing length hypothesis so that it is applicable over the entire region of turbulent shear flow including the points of maximum or minimum velocity. It is interesting to note in advance that von Kármán's similarity hypothesis is obtained as a special case of the present model.

REFORMULATION OF PRANDTL'S MIXING LENGTH THEORY

Consider the simplest case of incompressible parallel flow in which the velocity varies only from streamline to streamline. The principal direction of flow is taken to be parallel to the x axis, and we have

$$\bar{u} = \bar{u}(y); \quad \bar{v} = 0; \quad \bar{w} = 0$$

Now we make the first assumption mentioned above, that the fluid lumps retain their identity over a certain distance and then mix with the surroundings. Thus, when a mass of fluid traverses a distance l in the positive y direction (see Figure 1), the change in velocity is given by

$$|\Delta u_1| = |\bar{u}(y) - \bar{u}(y-l)| \approx \left| l\frac{d\bar{u}}{dy} - \frac{l^2}{2}\frac{d^2\bar{u}}{dy^2} \right| \tag{5}$$

Similarly, a lump of fluid traversing a distance l in the negative y direction will undergo the change in velocity given by

$$-\frac{\overline{u'v'}}{u^{*2}} = \begin{cases} c l_1^2 \left(\frac{du^+}{dy^+}\right)^2, & \left| l_1 \frac{du^+}{dy^+} \right| \geq \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \quad (14a) \\ \frac{c}{4} l_1^4 \left(\frac{d^2u^+}{dy^{+2}}\right)^2, & \left| \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| \geq \left| l_1 \frac{du^+}{dy^+} \right| \quad (14b) \end{cases}$$

$$|\Delta u_2| = |\overline{u}(y+l) - \overline{u}(y)| \approx \left| l \frac{d\overline{u}}{dy} + \frac{l^2}{2} \frac{d^2\overline{u}}{dy^2} \right| \quad (6)$$

where l is positive in both cases.

In Equations (5) and (6) we have expanded the velocity $\overline{u}(y)$ in a Taylor series in the positive and negative directions and neglected terms containing third- and higher-order derivatives. The point of departure of the present approach is that second-order derivatives in the Taylor series are retained. It is expected that these terms may become important when the velocity gradient is small and, consequently, when Equation (4) is not satisfied. It is assumed in Equations (5) and (6) that

$$l << 3 \left| \frac{d^2\overline{u}/dy^2}{d^3\overline{u}/dy^3} \right| \quad (7)$$

Let us now assume that the time average of the absolute value of the fluctuation caused by the velocity differences Δu_1 and Δu_2 can be obtained as

$$\overline{|u'|} = \frac{1}{2}(|\Delta u_1| + |\Delta u_2|) = \frac{1}{2}\left[\left| l \frac{d\overline{u}}{dy} - \frac{l^2}{2} \frac{d^2\overline{u}}{dy^2} \right| + \left| l \frac{d\overline{u}}{dy} + \frac{l^2}{2} \frac{d^2\overline{u}}{dy^2} \right| \right] \quad (8)$$

Equation (8) can be written in terms of dimensionless quantities

$$\frac{\overline{|u'|}}{u^*} = \frac{1}{2}\left[\left| l_1 \frac{du^+}{dy^+} - \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| + \left| l_1 \frac{du^+}{dy^+} + \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| \right] \quad (9)$$

Let us assume with Prandtl that the transverse component v' is proportional to u', so that

$$\overline{|v'|} = c_1 \overline{|u'|} \quad (10)$$

The correlation coefficient c_2 can be defined as

$$-\overline{u'v'} = c_2 \overline{|u'|}\,\overline{|v'|} \quad (11)$$

Combining Equations (9), (10), and (11), we obtain

$$-\frac{\overline{u'v'}}{u^{*2}} = \frac{c}{4}\left[\left| l_1 \frac{du^+}{dy^+} - \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| + \left| l_1 \frac{du^+}{dy^+} + \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| \right]^2 \quad (12)$$

where

$$c = c_1 c_2 \quad (13)$$

Since it is known that

$$\frac{1}{4}[|a-b| + |a+b|]^2 = \begin{cases} a^2, & |a| \geq |b| \\ b^2, & |b| \geq |a| \end{cases}$$

Equation (12) reduces exactly to

It is interesting to observe that when Equation (4) is satisfied, we get the result obtained by Prandtl where the constant c is absorbed in the mixing length so that $l_p = l_1\sqrt{c}$. However, when Equation (4) is not satisfied but Equation (7) applies, the present formulation suggests Equation (14b) for the Reynolds stress. Another interesting point emerging from the present analysis is that Equations (14a) and (14b) will intersect when

$$\left| l_1 \frac{du^+}{dy^+} \right| = \left| \frac{l_1^2}{2} \frac{d^2u^+}{dy^{+2}} \right| \quad (15)$$

or when

$$\frac{l_p u^*}{\nu} = l_1\sqrt{c} = k \left| \frac{du^+/dy^+}{d^2u^+/dy^{+2}} \right| \quad (15a)$$

where k $(= 2\sqrt{c})$ is von Kármán's universal constant. Equation (15) is identical to the mixing length expression proposed by von Kármán on the basis of his similarity hypothesis. Thus it is clear that Equation (12) or its equivalent, Equation (14), combines Prandtl's mixing length theory and von Kármán's similarity hypothesis into a simple unified form. That is, their results are both special cases of a more general phenomenological theory.

It must be emphasized again that the first assumption mentioned previously still applies to the present approach. Nevertheless, we believe that Equation (14) will prove to be more useful from the engineering point of view. To illustrate this point let us consider the case of fully developed turbulent flow in a pipe, where the shear stress distribution is given by

$$\frac{\tau}{\tau_w} = 1 - \frac{y^+}{y_m^+} = \frac{du^+}{dy^+} - \frac{\overline{u'v'}}{u^{*2}} \quad (16)$$

In the region near the wall, Equation (14a) will apply, and assuming that

$$-\frac{\overline{u'v'}}{u^{*2}} >> \frac{du^+}{dy^+}$$

we get

$$-\frac{\overline{u'v'}}{u^{*2}} = c l_1^2 \left(\frac{du^+}{dy^+}\right)^2 \approx 1 - \frac{y^+}{y_m^+} \quad (17)$$

Prandtl assumed a linear relationship for l_p, $l_p = ky$. However, from the experiments of Reichardt (*10*), we know that the eddy diffusivity, and hence the mixing length, increases as we move away from the wall and reaches a maximum somewhere between $y^+/y_m^+ = 0.4$ and 0.6. One of the simplest ways to approximate this is to express the mixing length as

$$\sqrt{c}\, l_1 = \left(\frac{l_p u^*}{\nu}\right) = ky^+\left(1 - \frac{y^+}{y_m^+}\right) \quad (18)$$

It can be seen that l_1 has a maximum at $y^+/y_m^+ = 0.5$. Substituting Equation (18) in (17) and integrating, we get

$$u^+ = \frac{1}{k} \ln \left\{ \frac{4y_m^+ \left(1 - \sqrt{1 - \frac{y^+}{y_m^+}}\right)}{1 + \sqrt{1 - \frac{y^+}{y_m^+}}} \right\} + B \tag{19}$$

where B is the integration constant

In the region close to the center of the pipe, Equation (14b) will apply, so that

$$-\frac{\overline{u'v'}}{u^{*2}} = \frac{c}{4} l_1^4 \left(\frac{d^2u^+}{dy^{+2}}\right)^2 \approx 1 - \frac{y^+}{y_m^+} \tag{20}$$

The correlation coefficient c_2 is zero at the center because of symmetry. Also, it is known from hot wire anemometer experiments (*11*) that the correlation coefficient is zero at the center and increases approximately linearly as one moves toward the wall, so that we can write

$$c_2 \sim 1 - \frac{y^+}{y_m^+}$$

Since for the region near the center c_1 is very nearly constant (*11*), we have

$$c = c_1 c_2 \sim 1 - \frac{y^+}{y_m^+}$$

In the central portion we have large eddies the size of which is on the same order as the radius of the pipe (*2*, *3*), and consequently $l_1 \sim y_m^+$. Combining these results, we get

$$\frac{c}{4} l_1^4 = \frac{y_m^{+4}}{4A^2} \left(1 - \frac{y^+}{y_m^+}\right) \tag{21}$$

where A is an unknown constant. From (20) and (21) we then obtain, upon integration

$$u_m^+ - u^+ = A \left(1 - \frac{y^+}{y_m^+}\right)^2 \tag{22}$$

Equating u^+, du^+/dy^+, and d^2u^+/dy^{+2} as calculated from Equations (19) and (22) evaluated at the intersection $y^+ = y_i^+$, we obtain

$$y_i^+/y_m^+ = 0.4 \tag{23}$$

$$A = \frac{2.7}{k} \tag{24}$$

and

$$u_m^+ = \frac{1}{k} \ln 1.35\, y_m^+ + B \tag{25}$$

If we still define the coefficient of eddy diffusion by Equation (2), we have

$$\left(\frac{\epsilon}{\nu}\right) = k\, y^+ \left(1 - \frac{y^+}{y_m^+}\right)^{3/2}, \quad \frac{y^+}{y_m^+} \leq 0.4 \tag{26a}$$

$$\left(\frac{\epsilon}{\nu}\right) = \frac{y_m^+}{2A}, \quad \frac{y^+}{y_m^+} \geq 0.4 \tag{26b}$$

It is interesting to note that the assumption of a constant eddy diffusion coefficient and a parabolic profile for the outer layer are suggested by Hinze (*3*), ($A = 7.14$), and by Brodkey (*2*), ($A = 6$), for pipe flow. Taking $k = 0.4$ we have from Equation (24) $A = 6.75$. Recent experimental measurements of Brinkworth and Smith (*12*) indicate that $k = 0.377$ (or $1/k = 2.65$). Consequently, from Equation (24) we have $A = 7.155$ which is in very close agreement with the value 7.14 proposed by Hinze (*3*) on the basis of Laufer's data (*11*).

We will now compare the velocity profile and the eddy diffusivity expressions developed here with one of the most reliable empirical expressions due to Reichardt (*10*). Based on his experimental data, Reichardt proposed the following expression for the eddy diffusivity:

$$\frac{\epsilon}{\nu} = \frac{ky^+}{6} \left(2 - \frac{y^+}{y_m^+}\right) \left[1 + 2\left(1 - \frac{y^+}{y_m^+}\right)^2\right] \tag{27}$$

Now, since $-\overline{u'v'}/u^{*2} = (\epsilon/\nu)\, du^+/dy^+$, and by neglecting the viscous contribution to the shear stress, Equation (16) can be integrated to give

$$u^+ = \frac{1}{k} \ln \left[\frac{1.5y^+ \left(2 - \frac{y^+}{y_m^+}\right)}{1 + 2\left(1 - \frac{y^+}{y_m^+}\right)^2} \right] + B \tag{28}$$

For small y^+, Equations (19) and (28) reduce to the familiar logarithmic law of the wall expression given by

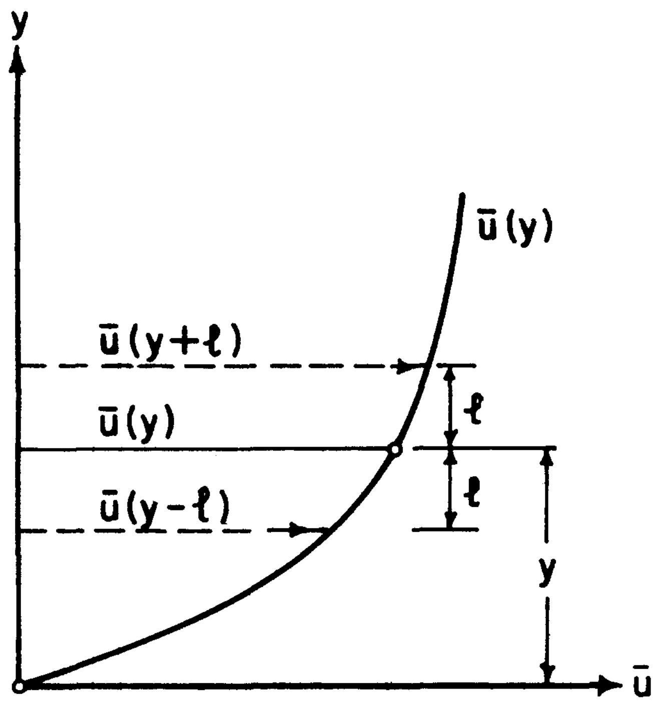

Fig. 1. Explanation of the mixing length concept.

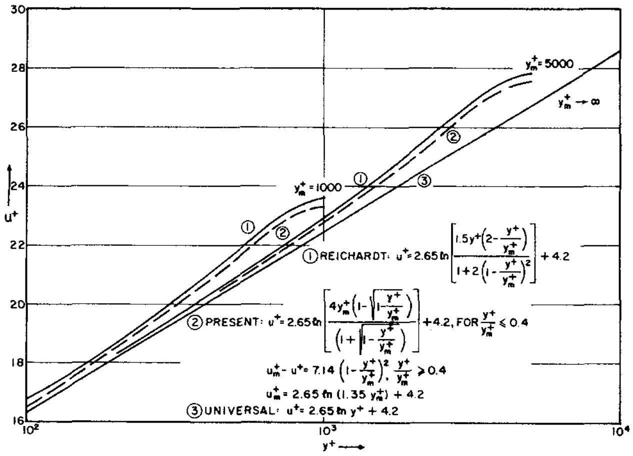

Fig. 2. Turbulent flow in a pipe: velocity distribution.

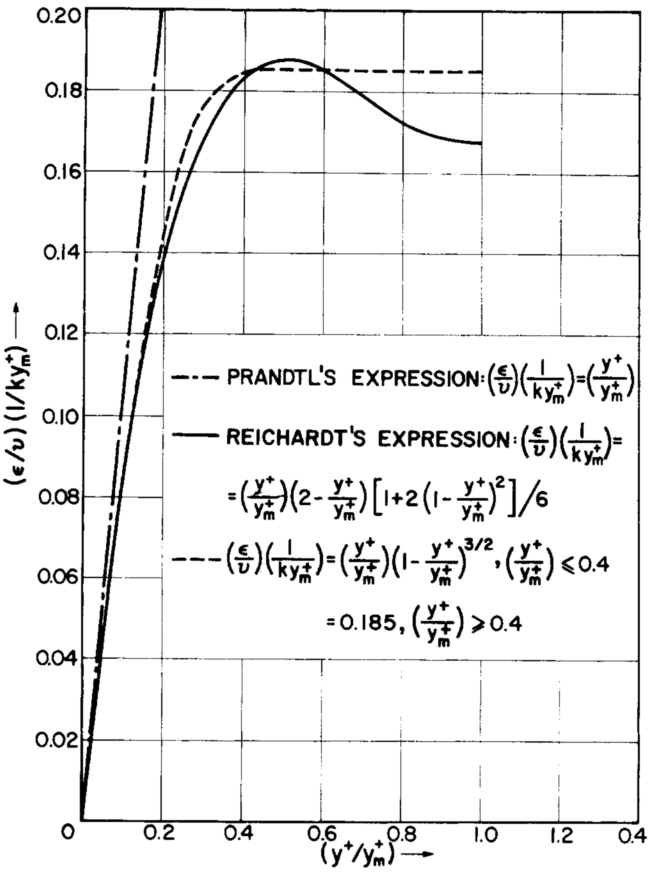

Fig. 3. Variation of eddy viscosity across pipe radius.

$$u^+ = \frac{1}{k}\ln y^+ + B \tag{29}$$

Recently, Brinkworth and Smith (*12*) used their experimental data to fit the Reichardt velocity profile, Equation (28), and proposed $k = 0.378$ and $B = 4.2$. Equation (28) with these values of the constants and Equations (19), (22), and (29) are compared in Figure 2. It can be seen that the agreement between Reichardt's expression and the expressions obtained here, Equations (19) and (22), is very good. The corresponding eddy diffusivity expressions are compared in Figure 3. Here we have selected the ordinate as $(\epsilon/\nu)/ky_m{}^+$, so that the curves become independent of the constant k and the Reynolds number. From Figure 3 we can see that the agreement between the present model and Reichardt's expression is fairly good.

We have given the example of turbulent flow in a pipe. By choosing proper expressions for the mixing length l_1 and the coefficient c, it was possible to obtain the law of the wall, Equation (19), and the velocity defect law, Equation (22), which are found to be in satisfactory agreement with the empirical equation of Reichardt.

CONCLUSIONS

Without significantly complicating the mathematical relations involved, the present approach enables one to eliminate the physically untenable aspect of mixing length theory which implies that eddy viscosity vanishes in the core of turbulent conduit flows if the velocity gradient vanishes. Furthermore, it is shown that the classical results obtained independently, and without an obvious connection between them, by Prandtl and von Kármán are special cases of a generalized notion of mixing length theory and are consistent with one another. The central region, or turbulent core, of conduit flows can also be analyzed in a consistent way by the present method.

ACKNOWLEDGMENT

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NOTATION

A = constant, see Equation (21)
B = constant, see Equation (19)
c = $c_1 \cdot c_2$
c_1 = constant, see Equation (10)
c_2 = correlation coefficient defined in Equation (11)
k = constant, see Equations (15) and (18)
l = mixing length, see Figure 1
l_1 = lu^*/ν
l_p = Prandtl's mixing length, see Equation (1)
l_{p1} = length parameter in Prandtl's modification
N_{Re} = Reynolds number, = $2y_m{}^+ u_B{}^+$ for tubes
R = radius of the pipe
$\bar{u}$ = time average axial velocity
u' = fluctuating velocity in axial direction
u^* = $\sqrt{\tau_w/\rho}$, friction velocity
u^+ = $\bar{u}/u^*$
$u_B{}^+$ = dimensionless area average velocity
$u_m{}^+$ = u^+ at $y^+ = y_m{}^+$
$\bar{v}$ = time average transverse velocity
v' = fluctuating velocity in transverse direction
$\bar{w}$ = time average velocity in z direction
y = distance measured from the boundary in a direction normal to the boundary
y^+ = yu^*/ν
$y_i{}^+$ = point of intersection of Equations (19) and (22)
$y_m{}^+$ = Ru^*/ν

Greek Letters

ρ = density
μ = molecular viscosity
ϵ = eddy diffusion coefficient, see Equation (2)
ν = μ/ρ, kinematic viscosity
τ = shear stress
τ_w = shear stress at the wall

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